# Introduction to Geometric Modeling

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Ship Design I

MSc in Marine Engineering and Naval Architecture

## Summary

1. Parametric Curves
2. Parametric Surfaces

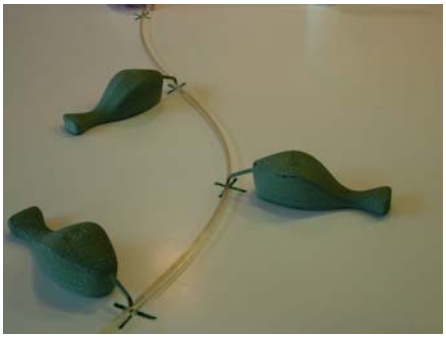
## Parametric Curves

1. Mathematical Formulations
   - Cubic Splines
   - Bézier
   - B-Spline
   - Beta-Spline
   - NURBS
2. Interpolation and approximation of curves
3. Analysis of curves



## Cubic Spline (1)

Considering the wooden spline (*virote*) a thin elastic beam, and for small deflections, the Euler law relates the deflection of the beam axis y(x) with the bending moment M(x) by the expression:



$$y''(x) = \frac{M(x)}{EI}$$

where:

E – Modulus of Young

I – Moment of inertia of the beam section

## Cubic Spline (2)

- Assuming that the beam is simply supported on the weights, then the bending moment varies linearly between them, i.e., M(x) = Ax + B. Replacing in the expression and integrating results

$$y(x) = \iint \frac{M(x)}{EI} dx = \frac{1}{\mathrm{EI}} \iint (Ax + B) dx = Ax^3 + Bx^2 + Cx + D$$

In each segment, the curve can be defined as a function of the parameter t normalized for the interval [0,1]

$$P(t) = At^3 + Bt^2 + Ct + D$$

The constants can be obtained from the following boundary conditions:

$$\begin{cases} P(0) = p_0 \\ P(1) = p_1 \\ P'(0) = T_0 \\ P'(1) = T_1 \end{cases}$$



## Cubic Spline (3)

Finally the curve can be represented in the matrix form as

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} [H][G]$$

where

$$[H] = \begin{bmatrix} +2 & -2 & +1 & +1 \\ -3 & +3 & -2 & -1 \\ 0 & 0 & +1 & 0 \\ +1 & 0 & 0 & 0 \end{bmatrix} \qquad [G] = \begin{bmatrix} p_i \\ p_{i+1} \\ T_i \\ T_{i+1} \end{bmatrix}$$

## Bézier Curves (1)

- The curves generally known as Bézier resulted from separate research from Casteljau (Citroen) and Pierre Bézier (Renault) in the beginning of the 1960s.
- A Bézier curve is defined by:

$$P(t) = \sum_{i=0}^{n} C_i B_{n,i}(t) \qquad for \;\; 0 \le t \le 1$$

where $B_{n,j}$ are the Bernestein base functions, of degree *n*

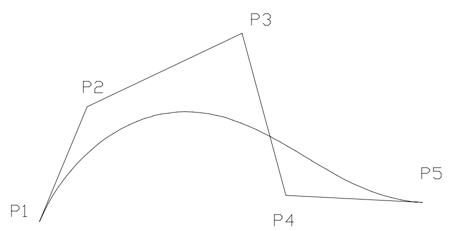
$$B_{n,i} = \frac{n!}{i!(n-i)!}(1-t)^{n-i} t^i$$

$$= \binom{n}{i}(1-t)^{n-i} t^i \qquad \text{for } i = 0,\ 1,\ \ldots,\ n$$



## Bézier Curves (2)

- The Bézier curve is tangent to the first and last segments of the control polygon
- The curve order is equal to the number of vertices of the control polygon.
- The curve is entirely contained in the convex hull of the control points.



## B-Spline Curves (1)

- They were studied by N. Lobatchevsky in the XIX century
- Their use for curve fitting to experimental data began in 1946 with Schoenberg
- They were first introduced in CAD systems by J. Ferguson (Boeing) in 1963.

$$C_k(t) = \sum_{i=0}^{n} P_i N_{i,k}(t)$$

Where $C_i$ are the points of the control polygon and $N_{i,k}$ are the B-Spline base functions, of order k, that can be computed by the recursive expression from Cox/de Boor:

$$N_{i,0}(t) = 1 \quad para \quad t_i \le t < t_{i+1}$$
$$= 0$$
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Defined over a knot vector

$$X = \{t_1, t_2, t_3, \ldots, t_m\}$$



## B-Spline Curves (2)

- The knot vector is a non-decreasing sequence of numbers
- The knot vector can be classified as:
  - Uniform – the increment between knots is constant
    { 0.0, 0.5, 1.0, 1.5, 2.0 }
  - Periodic – the increment is constant and equal to 1
    { 0, 1, 2, 3, 4, 5 }
  - Non-Periodic – the increment of the interior knots constant and equal to 1 and the knots of the extremities with multiplicity equal to the order
    { 0, 0, 0, 1, 2, 3, 4, 5, 5, 5 }
  - Non-Uniform - the increment of the interior knots not necessarily constant and the knots of the extremities with multiplicity equal to the order
    { 0, 0, 0, 1.0, 1.4, 2.0, 2.3, 3.0, 3.0, 3.0 }

## B-Spline Curves (3)

The B-Spline curves have the following properties:

- Linear precision
- Convex hull, in k consecutive control points
- Variation diminishing
- Are invariant when submitted to affine transformations
- When the order of the B-Spline is equal to the number of control points, the knot vector consists only in the values of the extremities with the multiplicity equal to the order

  { 0, 0, 0, 0, 1, 1, 1, 1 }

  and the B-Spline base functions are equivalent to Bernestein functions and the curve degenerates into a Bézier curve.



## Beta-Splines (1)

- Os cubic Beta-splines were introduced on 1981 by Barsky
- They are a generalization of the B-Splines based in notions of geometric continuity and in the mathematical modeling of tension
- The requirements of parametric continuity of the 2ª order ($C^2$) between the B-Splines segments is replaced by the requirements of geometric continuity of 2ª order ($G^2$) of the unit tangent vector and of the curvature vector
- This originates discontinuities of the 1st and 2nd parametric derivative, that are expressed as functions of the parameters β1 and β2, designated by bias and tension, respectively.

# Beta-Splines (2)

- A Beta-spline curve is defined by:

$$C_i(u) = \sum_{r=-2}^{1} b_r\left(\beta_1, \beta_2; u\right) P_{i+r} \qquad p/\, 0 \le \mathrm{i} < 1$$

where $b_\gamma$ are the base functions

$$b_r(\beta_1, \beta_2; u) = \sum_{g=0}^{3} c_{gr}(\beta_1, \beta_2) u^g \qquad p/\, 0 \le \mathrm{u} < 1 \,\mathrm{e}\;\; r = -2, -1, 0, 1$$

- The parametric continuity reflects the fair variation of the parameterization and not necessarily of the curve
- The geometric continuity is a measure of the continuity that is independent from the parameterization



# NURBS Curves

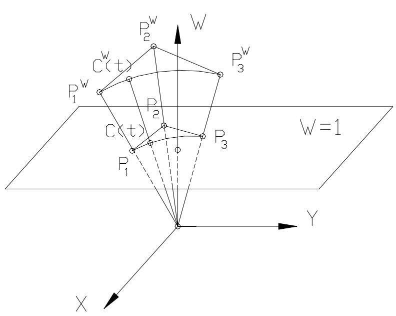
$$C(u) = \frac{\sum_{i=0}^{n} P_i \cdot w_i \cdot N_{i,p}(u)}{\sum_{i=0}^{n} w_i \cdot N_{i,p}(u)}$$

$$N_{i,0}(u) = 1 \quad \text{para} \quad u_i \le \mathrm{u} < u_{i+1}$$

$$= 0$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

$$U = \left\{0, 0, \ldots, 0, u_{k+1}, u_{k+2}, \ldots, u_n, u_{n+1}, \ldots, u_{n+k}\right\}$$



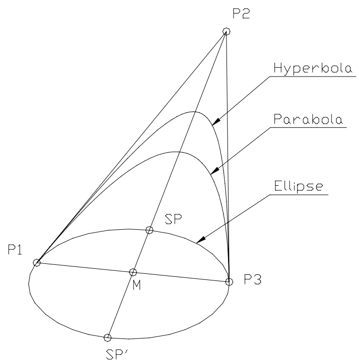
# Representation of Conic Shapes (1)

- A NURBS curve of the 2nd degree, with 3 points represents a conic shape if the conic form factor, $k_c$, defined by:

$$k_c = \frac{w_1 . w_3}{4. w_2^2}$$

Has one of the following values

$$4k_c < 1.0 \Rightarrow elipse$$
$$4k_c = 1.0 \Rightarrow parabola$$
$$4k_c > 1.0 \Rightarrow hiperbole$$





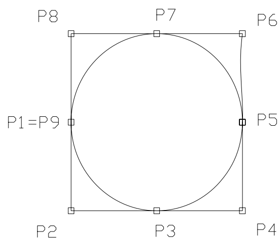

# Representation of Conic Shapes (2)

- To represent a circular arc, the 3 control points [P1, P2, P3] must be over the vertices of a triangle isosceles
- The arc radius obtained is computed by:

$$R = \frac{\sqrt{(1+4b^2)}}{4b} \qquad \text{where:} \qquad b = \frac{\sqrt{k^2-1}}{2}$$

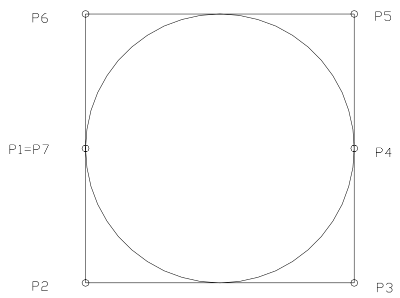
- Complete circumferences can be represented joining arcs
- With 9 points, 4 arcs of 90° can be joined

$$X = \{0,0,0,0.25,0.25,0.5,0.5,0.75,0.75,1.0,1.0,1.0\}$$
$$W = \{1.0, \sqrt{2}/2, 1.0, \sqrt{2}/2, 1.0, \sqrt{2}/2, 1.0, \sqrt{2}/2, 1.0\}$$
$$P = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9\}$$



## Representation of Conic Shapes (3)

- The previous representation can be simplified, removing the repeated knots 0.25 and 0.75
- The result is a circumference represented by only 7 control points

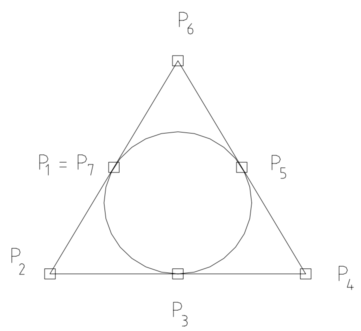



$$X = \{0,0,0,0.25,0.5,0.5,0.75,1.0,1.0,1.0\}$$
$$W = \{1.0,0.5,0.5,1.0,0.5,0.5,1.0\}$$
$$P = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$$



## Representation of Conic Shapes (4)

A circumference can also be obtained joining 3 arcs of 120°, defined by 7 control points.

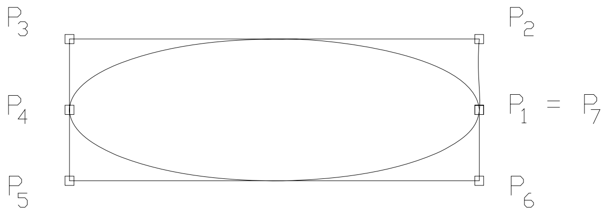



$$X = \left\{0,0,0,\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3},1.0,1.0,1.0\right\}$$
$$W = \{1.0,0.5,1,0.5,1.0,0.5,1.0\}$$
$$P = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

# Representation of Conic Shapes (5)

- A complete ellipse can be represented applying an affine transformation to a circumference, for example, one represented by 7 control points, keeping the distribution of the weights and the knot vector.

$$X = \{0.0, 0.0, 0.0, 0.25, 0.5, 0.5, 0.75, 1.0, 1.0, 1.0\}$$
$$W = \{1.0, 0.5, 0.5, 1.0, 0.5, 0.5, 1.0\}$$
$$P = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$$






# Summary - Parametric Curves (1)

| | **Advantages** | **Disadvantages** | **Obs.** |
|---|---|---|---|
| **Cubic Spline** | Interpolates data points | Can present unexpected inflections | |
| **Bézier** | The control polygon lies outside the data points | Global behavior<br>Degree increases directly with the increasing number of control points | |

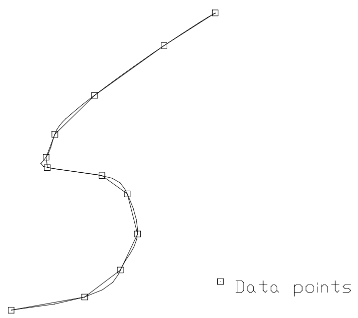
# Summary - Parametric Curves (2)

| | Advantages | Disvantages | Obs. |
|---|---|---|---|
| **B-Spline** | Local behavior<br>Degree independent of the number of control points | Can NOT represent conic shapes accurately | |
| **Beta-Spline** | Two additional parameters to control (bias and tension) | | Used in fairing methods |
| **NURBS** | Accurate representation of conics | It is difficult to take advantage of the additional coordinate (weight) | State of the art.<br>Used in most existing CAD systems |

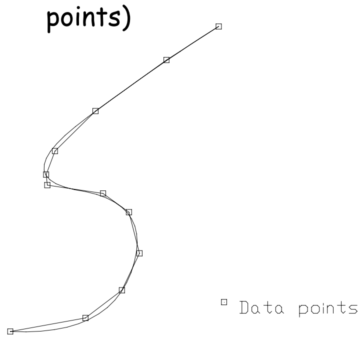


# Curve Generation

**Interpolation** (curve contains all the data points)




**Approximation** (curve tries to minimize the distance to all the data points)



## Analysis of Curve Curvature (1)

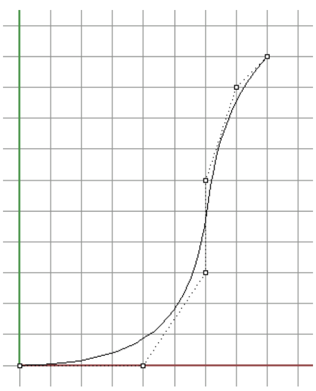
- The curvature of a space curve is defined by:

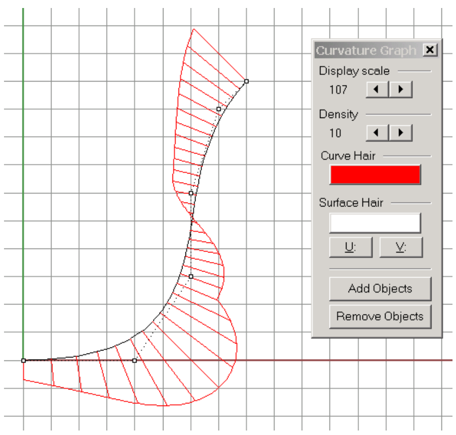
$$\kappa(t) = \frac{|x'(t) \times x''(t)|}{|x'(t)|^3}$$

- The distribution of this curvature along the curve can be represented using the method of the "porcupine"
  - vectors with modules proportional to the values of the curvature at each point
  - normal to the curve at that point
  - oriented to the opposite side of the centre of curvature



## Analysis of Curve Curvature (2)





# Parametric Surfaces

1. NURBS Surfaces
2. Surfaces Generation
   - Extrusion
   - Lofting
   - Sweeping
   - Revolution
   - Grid Interpolation
   - Primitives
3. Surface Analysis
   - Shading
   - Contours
   - Curvatures
   - Isophotes
   - Reflection Lines



# NURBS Surfaces

- A NURBS surface of degree (k,l) in the directions (u,v) is defined by the expression:

$$S(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}P_{i,j}.w_{i,j}.N_{i,k}(u)M_{j,l}(v)}{\sum_{i=0}^{n}\sum_{j=0}^{m}w_{i,j}.N_{i,k}(u)M_{j,l}(v)}$$

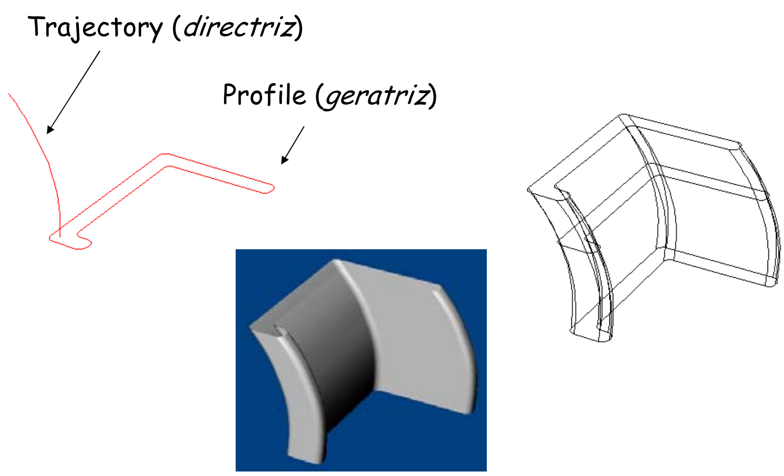
$$N_{i,0}(u)=1 \quad p/ \quad u_i \le u < u_{i+1}$$
$$=0$$

$$N_{i,k}(u)=\frac{u-u_i}{u_{i+k}-u_i}N_{i,k-1}(u)+\frac{u_{i+k}-u}{u_{i+k}-u_{i+1}}N_{i+1,k-1}(u)$$

$$M_{j,0}(v)=1 \quad p/ \quad v_j \le v < v_{j+1}$$
$$=0$$

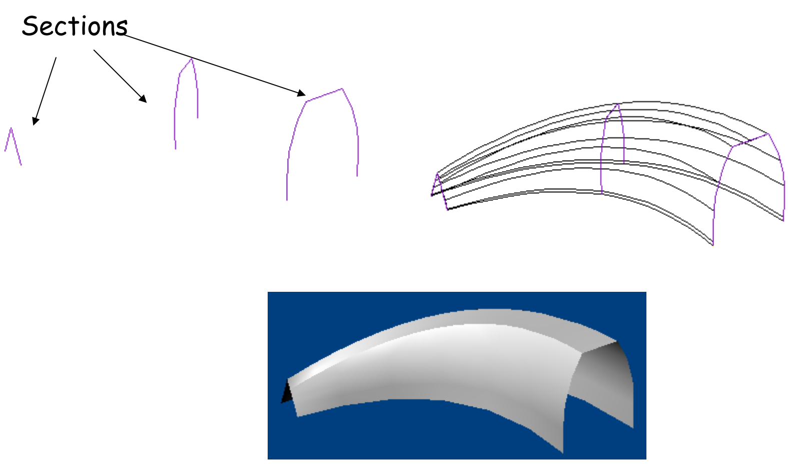
$$M_{j,l}(v)=\frac{v-v_j}{v_{j+l}-v_j}M_{j,l-1}(v)+\frac{v_{j+l}-v}{v_{j+l}-v_{j+1}}M_{j+1,l-1}(v)$$

# Extrusion






# Lofting



## Lofting in Shipbuilding

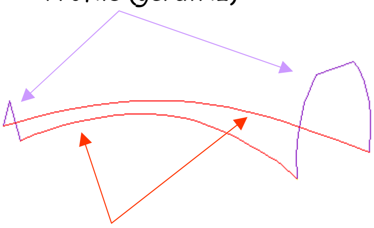
- The designation *lofting* has origin in shipbuilding
- Designates the development of the ship hull surface interpolating the shape of a set of cross sections, that was carried out in the loft room (*sala do risco*)

The primitive building process was similar to the modeling process, using the frames to shape the hull surface form.

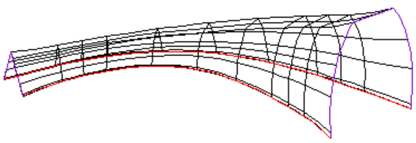


## Sweeping

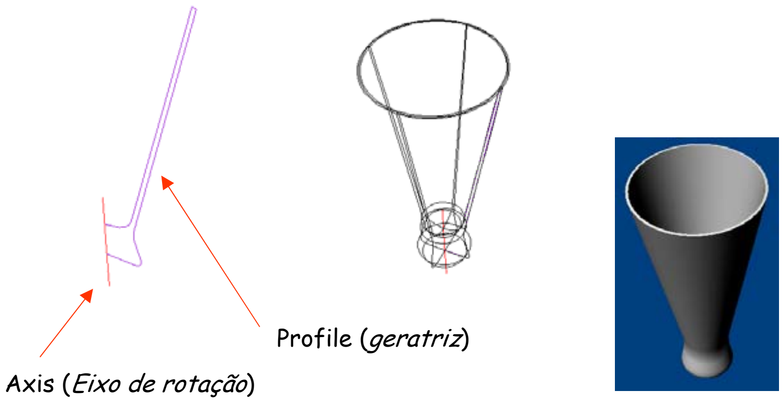
Profile (*geratriz*)



Trajectory (*directriz*)



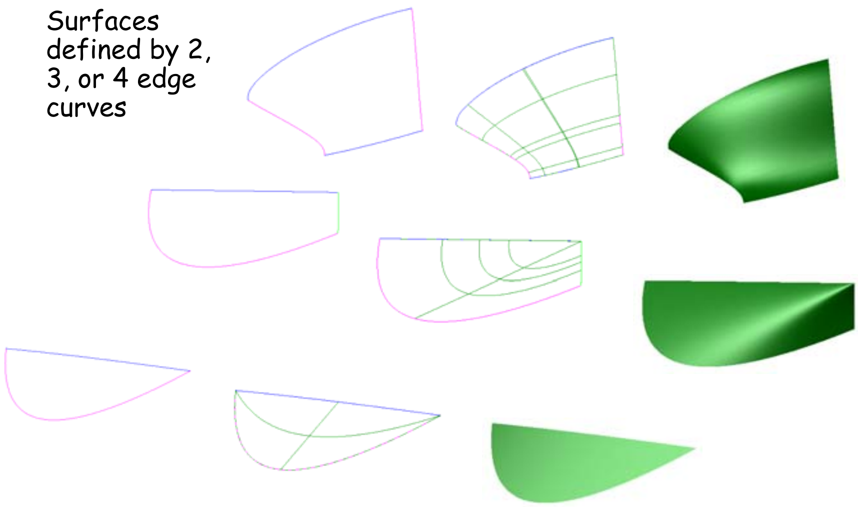
# Surfaces of Revolution





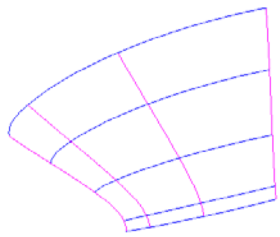

# Edge Curves

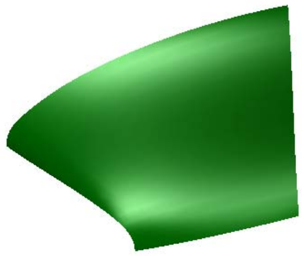
Surfaces defined by 2, 3, or 4 edge curves



## Grid Interpolation

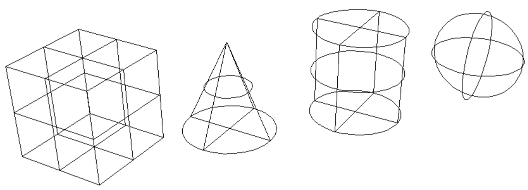
- Surface generated from a regular grid of curves
- Provide a better control over the inner shape of the surface



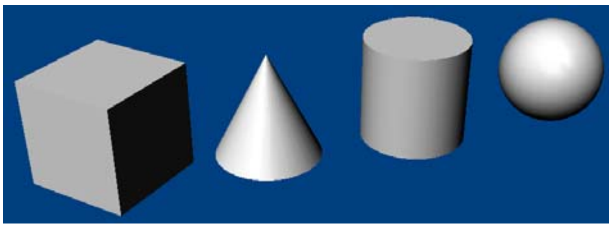




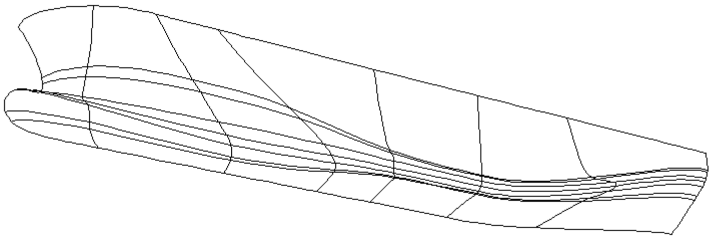
## Elementary Primitive Shapes



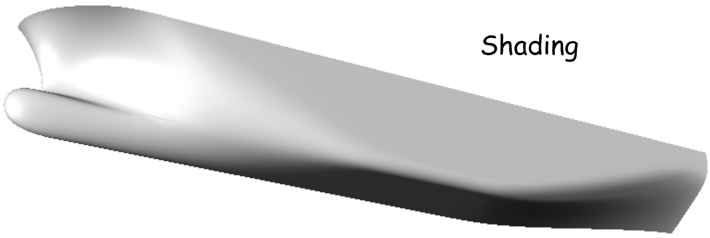
- Box
- Cone
- Cylinder
- Sphere



## Surface Analysis - Shading



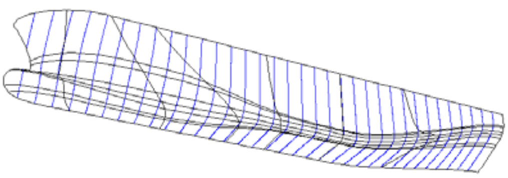
Shading

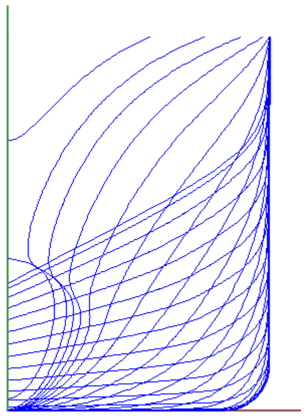




## Surface Analysis - Contours

Contours

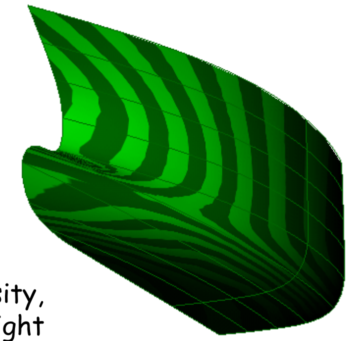


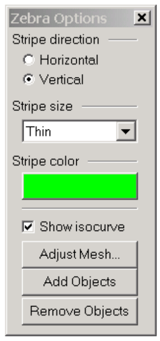


## Surface Analysis - Isophotes

Isophotes:

Analyze/Surface/Zebra






Lines of constant light intensity, created by a set of parallel light sources, with a given direction, L

$$n \cdot L = \cos\alpha$$



## Surface Analysis – Curvature

**Mean Curvature**

$$H = \frac{2FM - (EN + GL)}{2(EG - F^2)}$$

**Gauss Curvature**

$$K = \frac{LN - M^2}{EG - F^2}$$

**1st Fundamental Form Coefficients**

$$E = r_u . r_u \quad F = r_u . r_v \quad G = r_v . r_v$$

**2nd Fundamental Form Coefficients**

$$L = n.r_u \quad M = n.r_{uv} \quad N = n.r_{vv}$$

Curvatures expressed as a function of the max. and min. curvatures

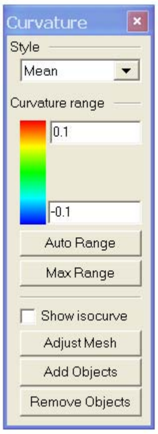
$$H = \frac{1}{2}(\kappa_{\min} + \kappa_{\max})$$

$$K = \kappa_{\min}\,\kappa_{\max}$$

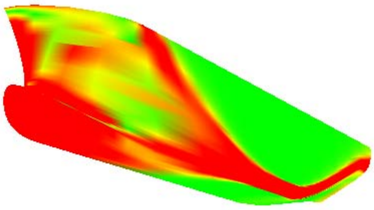
Surface normal unit vector

$$n = \frac{r_u \times r_v}{|r_u \times r_v|} \quad \text{p/}\ |r_u \times r_v| \neq 0$$

# Mean Curvature

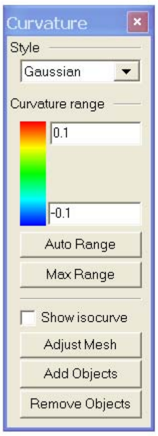



Mean Curvature distribution

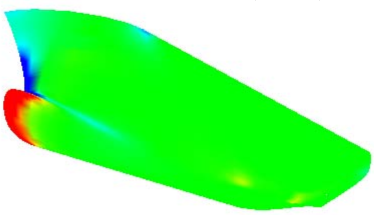




# Gauss Curvature




Gauss Curvature distribution



| | |
|---|---|
| K < 0 | Surface with double curvature (saddle shape) |
| K = 0 | Developable surface |
| K > 0 | Surface with single curvature (concave or convex) |